Albert-Ludwigs-Universität, Institut für Informatik
Prof. Dr. Fabian Kuhn
M. Ahmadi, P. Schneider, J. Uitto

January 25, 2018

Algorithm Theory - Winter Term 2017/2018
Exercise Sheet 7

Hand in by Monday 14:15, February 5, 2018

The points on this exercise sheet are bonus points. Earned points are added to your score but this sheet will not be considered in the maximum number of achievable points. Note the earlier deadline!

Exercise 1: Maximum Coverage (10+3 Points)

Let X be a set of n elements and let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a system of m subsets $S_1, \dots, S_m \subseteq X$. For an integer parameter $k \geq 1$, the maximum coverage problem asks for k sets $A_1, \dots, A_k \in \mathcal{S}$ such that $|\bigcup_{j=1}^{k} A_j|$ is maximized.

The following greedy approximation algorithm starts with $A = \emptyset$ and does k iterations. In iteration j it adds a subset $A_j \in \mathcal{S}$ to A (i.e. $A \leftarrow A \cup A_j$) that *maximizes* $|A \cup A_j|$ in the *current step*.

Let $O := \bigcup_{j=1}^{k} O_j$ with $O_1, \dots, O_k \in \mathcal{S}$ such that $|O|$ is *maximized overall*. Let $A^{(i)} = \bigcup_{j=1}^{i} A_j$, be the union of subsets chosen by the greedy algorithm until iteration i (i.e. $A^{(k)} = A$).

(a) Show that for any $i \in \{1, \dots, k\}$ it holds that $|O| - |A^{(i)}| \leq (1 - 1/k)^i |O|$.

(b) Show that the approximation ratio of the greedy algorithm is larger than $1 - \frac{1}{e}$.[1]

Exercise 2: Acyclic Graphs (10 Points)

Consider the following problem. Given a directed graph $G = (V, E)$, the goal is to determine a maximum cardinality set $E' \subseteq E$ such that the graph induced by E' is acyclic. Provide a $\frac{1}{2}$-approximation algorithm for this problem. Prove that the approximation ratio is at least $\frac{1}{2}$.

Hint: Considering any set of nodes arbitrarily partitioned into two sets A and B, the set of outgoing edges from A to B induce an acyclic graph as well as the set of outgoing edges from B to A.

[1] The constant e is Euler's number.

Exercise 3: Online Vertex Cover (9+4+4 Points)

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is called a *vertex cover* if and only if for every edge $\{u, v\} \in E$ at least one of its endpoints is in S. The minimum vertex cover problem is to find such a set S of minimum size.

We are considering the following online version of the minimum vertex cover problem. Initially, we are given the set of nodes V and an empty vertex cover $S = \emptyset$. Then, the edges appear one-by-one in an online fashion. When a new edge $\{u, v\}$ appears, the algorithm needs to guarantee that the edge is covered (i.e., if this is not already the case, at least one of the two nodes u and v needs to be added to S). Once a node is in S it cannot be removed from S.

(a) Provide a deterministic online algorithm with competitive ratio at most 2. That is, your online algorithm needs to guarantee at all times that the vertex cover S is at most by a factor 2 larger than a current optimal vertex cover. Prove the correctness of your algorithm.

(b) Show that any deterministic online algorithm for the online vertex cover problem has competitive ratio at least 2.

(c) Use Yao's principle to show that any randomized online algorithm for the online vertex cover problem has competitive ratio at least 3/2.